Closing Tue: *12.6, 13.1*
Closing Thu: *13.2*
Closing ***Sunday****:* *13.3*

13.1 Intro to Curves (continued)

Let's explore curves by looking at some intersections problems...

Three types of intersections

(a) ***("Easy") A curve and surface***.
Combine, solve for *t.*

(b) ***("Medium") Two curves***.
Use different parameters!
Combine, solve for both parameters.

(c) ***("Hard?")*** **Two surfaces.**
A typical goal is to try to *parameterize* the curve...
Combine conditions:
- Pick one variable as t
- Solve for others.
- And/or use circular motion.

Entry Task: *(13.1 / 7, 8)* *curve/surface*

Find the intersection of

$$x = t, y = \cos(\pi t), z = \sin(\pi t)$$

and

$$(x^2 - y^2 - z^2 = 3.)\cdot -1$$

$$-x^2 + y^2 + z^2 = -3$$

Hyperboloid of 2 sheets

$$y^2 + z^2 = 1$$

$$t^2 - \cos^2(\pi t) - \sin^2(\pi t) = 3$$

$$t^2 - 1 = 3$$

$$t^2 = 4$$

$$t = \pm 2$$

intersection points

$$\boxed{\begin{aligned}(x,y,z) &= (2,1,0)\\ (x,y,z) &= (-2,1,0)\end{aligned}}$$

Example: *(13.1/12, 13) two curves*

Sp'17 Exam Problem

Given:

$$\boldsymbol{r}_1(t) = \langle 2t, 3t^2, 2t^3 \rangle$$
$$\boldsymbol{r}_2(t) = \langle 2-2t, 3+3t, 2-6t \rangle$$

Find the (x,y,z) point(s) at which the **paths** of the two particles described cross.

① $2t = x = 2-2u \longrightarrow u = 1-t$

② $3t^2 = y = 3+3u \rightarrow t^2 = 1+u \rightarrow t^2 = 1+1-t$

③ $2t^3 = z = 2-6u$

$t^2 = 2-t$

$t^2 + t - 2 = 0$

$(t-1)(t+2) = 0$

$t = 1, -2$

$u = 1-1 \quad u = 1+2$

$u = 0, 3$

@ $t=1$, $u=0$

$x = -4$

$y = 12$

$z = 2$

@ $t = -2$, $u = 3$

$x = 2$

$y = 3$

$z = -16$

$\boxed{(-4, 12, 2) \text{ and } (2, 3, -16)}$

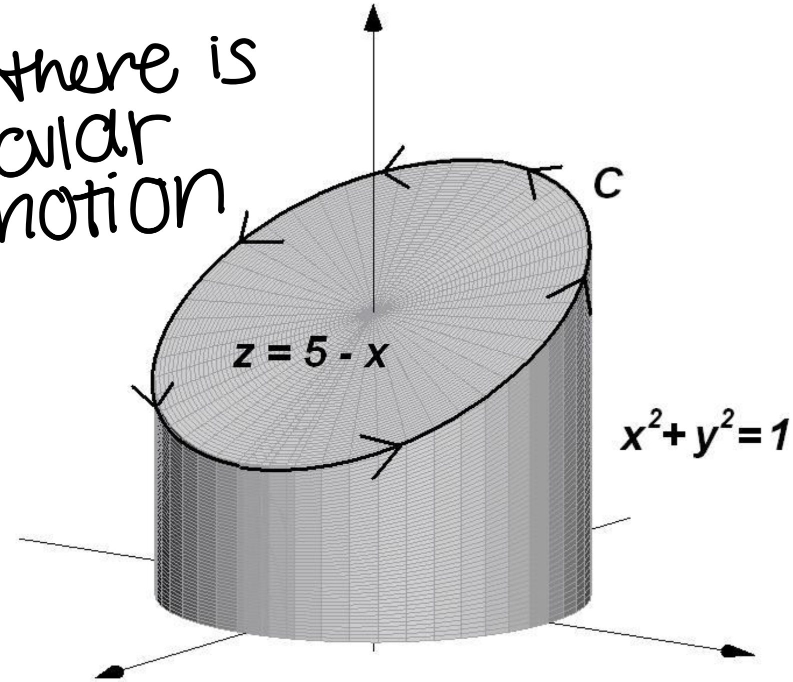
Visuals: https://www.math3d.org/NKOiXjHB infinate options

Example: *(13.1/9-11) surface/surface*

Find *any* parametric equations that describe the curve of intersection of

$$x^2 + y^2 = 1 \text{ and } z = 5 - x$$

if there is circular motion



Option 1

$z = t$

$t = 5 - x$

$x = 5 - t$

$y = \pm\sqrt{1-(5-t)^2}$

$$\boxed{\begin{aligned} x &= 5-t \\ y &= \pm\sqrt{1-(5-t)^2} \\ z &= t \end{aligned}}$$

Option 2

$$\boxed{\begin{aligned} x &= 1\cdot\cos(t) \\ y &= 1\cdot\sin(t) \\ z &= 5-\cos(t) \end{aligned}}$$

b/c $\cos^2(t) + \sin^2(t) = 1$

ex:

$x^2 + y^2 = 9 \rightarrow \begin{aligned} x &= 3\cos(t) \\ y &= 3\sin(t) \end{aligned}$

Visuals: https://www.math3d.org/ickrjljF

Section 13.2 – Vector Calculus Intro

Entry Task:

$\vec{r}(t) = \langle \overset{x}{t}, \overset{y}{\cos(2t)}, \overset{z}{\sin(2t)} \rangle$.

1. Find $\vec{r}'(t)$. ← tangent vector
2. Find $\vec{r}(\pi/4)$.
3. Find $\vec{r}'(\pi/4)$.

Brief Summary - calculus component-wise!

1st Deriv. vector: $\vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$

2nd Deriv. vector: $\vec{r}''(t) = \langle x''(t), y''(t), z''(t) \rangle$

Anti-deriv. vector:

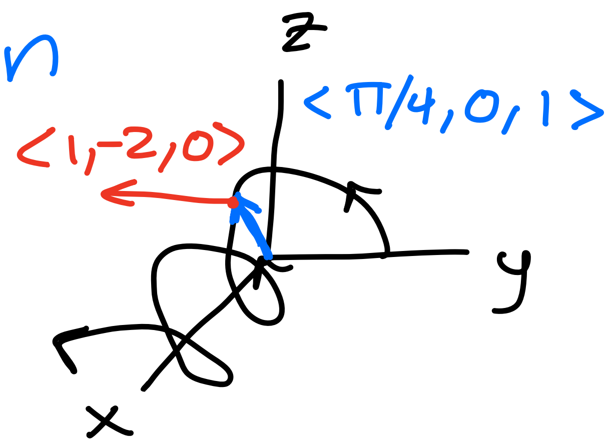
$$\int \vec{r}(t)dt = \left\langle \int x(t)dt, \int y(t)dt, \int z(t)dt \right\rangle$$

① $\vec{r}'(t) = < 1, -2\sin(2t), 2\cos(2t) >$

② $\vec{r}(\pi/4) = < \frac{\pi}{4}, 0, 1 >$ position

③ $\vec{r}'(\pi/4) = < 1, -2, 0 >$

direction



points to where I am

points to where I'm going

$$\lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} = \text{definition of derivative}$$

Visuals: https://www.math3d.org/pkXEorK4

A couple important follow-up facts

- $\vec{T}(t) = \frac{1}{|\vec{r}\,'(t)|}\vec{r}\,'(t)$ *unit tangent vector* (13.2/ 5, 6)

To find the tangent **line** to $\vec{r}(t)$ at $t = t_0$ (13.2 / 7)

- *Step 1*: Location $\vec{r}(t_0) = \langle x_0, y_0, z_0 \rangle$.
- *Step 2*: Direction vector $\vec{r}\,'(t_0) = \langle x'(t_0), y'(t_0), z'(t_0) \rangle$.

Example (continued)

$\vec{r}(t) = \langle t, \cos(2t), \sin(2t) \rangle$.

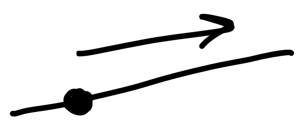
4. Find the unit tangent, $\vec{T}(t)$, at $t = \pi/4$.

$\vec{r}\,'(\pi/4) = \langle 1, -2, 0 \rangle$

$\vec{T}(t) = \frac{1}{\sqrt{1+4+0}} \langle 1, -2, 0 \rangle$

$\boxed{\vec{T}(t) = \left\langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}, 0 \right\rangle}$

5. Find parametric equation for the tangent line at $t = \pi/4$.



$$\boxed{\begin{aligned} x &= \tfrac{\pi}{4} + 1\,t \\ y &= 0 + -2\,t \\ z &= 1 + 0\,t \end{aligned}}$$

position vector = point

$(\pi/4, 0, 1)$

direction vector = $\vec{r}\,'(t)$

$\langle 1, -2, 0 \rangle$ direction

Example: (*13.2 / 7, 8, 9*)
W'15 Exam 1 - Loveless

$\vec{r}(t) = \langle 2t, 5, t^2 - 10t \rangle$ — 0 = hit xy-plane

$\vec{r}\,'(t) = \langle 2, 0, 2t - 10 \rangle$ — 0 = parallel xy-plane

↳ tangent

Consider the curve given by:

$$x = 2t, y = 5, z = t^2 - 10t$$

(a) There is one point on the curve at which the tangent line is parallel to the xy-plane. Find the tangent line at this point.

$2t - 10 = 0$

$t = 5$

$x = \quad + \quad t$

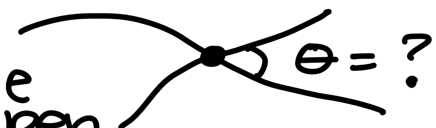
$y = \quad + \quad t$

$z = \quad + \quad t$

$r(5)$ $\quad$ $r'(5)$

(b) Find the angle of intersection of the original curve and this other curve

$$x = 7 + u, y = 2u + 11, z = u^2 + u - 22$$



angle between directions not positions

$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$

$\vec{r_1}\,'(t) = \langle 2, 0, 2t - 10 \rangle$

$\vec{r_2}(t) = \langle 1, 2, 2u + 1 \rangle$

$2t = 7 + u$

$5 = 2u + 11$

$t^2 - 10t = u^2 + u - 22$

$t = ?$ $\quad u = ?$

Visuals: https://www.math3d.org/ZgPOXWkh

Derivatives Quick Review

A basic example: Write down the derivative of

$$g(x) = x^3 \cos(2x) + \sqrt{1 + e^{3x}}$$

Calculus Fact Sheet

Essential Derivative Rules

$\frac{d}{dx}(x^n) = nx^{n-1}$	$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$	
$\frac{d}{dx}(e^x) = e^x$	$\frac{d}{dx}(b^x) = b^x \ln(b)$	
$\frac{d}{dx}(\sin(x)) = \cos(x)$	$\frac{d}{dx}(\tan(x)) = \sec^2(x)$	$\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$
$\frac{d}{dx}(\cos(x)) = -\sin(x)$	$\frac{d}{dx}(\cot(x)) = -\csc^2(x)$	$\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$
$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{x^2+1}$	$\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\left(\sec^{-1}(x)\right) = \frac{1}{x\sqrt{x^2-1}}$
$(FS)' = FS' + F'S$	$\left(\frac{N}{D}\right)' = \frac{DN' - ND'}{D^2}$	$[f(g(x))]' = f'(g(x))g'(x)$

From my Calculus 1 Fact Sheet:
math.washington.edu/~aloveles/Math126Materials/CalculusFactSheet2.pdf

Antiderivatives Quick Review

A basic example: Find $f(t)$, if

$$f'(t) = 6\sqrt{t} + \sin(t) + 9te^{t^2}$$

with $f(0) = 7$.

Essential Integral Rules

$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$	$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln\lvert ax+b\rvert + C$
$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$	$\int b^x\,dx = \frac{1}{\ln(b)}b^x + C$
$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + C$	$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + C$
$\int \sec^2(x)\,dx = \tan(x) + C$	$\int \csc^2(x)\,dx = -\cot(x) + C$
$\int \sec(x)\tan(x)\,dx = \sec(x) + C$	$\int \csc(x)\cot(x)\,dx = -\csc(x) + C$
$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$	$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$
$\int \tan(x)\,dx = \ln\lvert\sec(x)\rvert + C$	$\int \cot(x)\,dx = \ln\lvert\sin(x)\rvert + C$
$\int \sec(x)\,dx = \ln\lvert\sec(x)+\tan(x)\rvert + C$	$\int \csc(x)\,dx = \ln\lvert\csc(x)-\cot(x)\rvert + C$

$$\int \sec^3(x)\,dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln\lvert\sec(x)+\tan(x)\rvert + C$$

From my Calculus 1 Fact Sheet:
math.washington.edu/~aloveles/Math126Materials/CalculusFactSheet2.pdf

For Ch. 13, you MUST remember:

- finding "+C".
- **u-substitution** and **integration by parts**.

You won't need to remember any other integration methods in chapter 13 (but you will need to remember them in chapter 15)